Casimir force derived using the Euler-Maclaurin and the zeta function regularization techniques

N. B. Cook nigelcook@quantumfieldtheory.org

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Abstract

This paper derives Casimir force in detail using both the Euler-Maclaurin and the zeta function regularization techniques. Thus, we argue that there are different mathematical barrel organs you can use in physics, which deliver the same result. Going still further, you can even use rough approximations to formulate ideas before finding a mathematical proof, as Archimedes argued in *The Method* (T. L. Heath's translation): "Archimedes to Eratosthenes, greetings. I sent you on a former occasion some of the theorems discovered by me ... I thought fit to write out for you and explain in detail in the same book the peculiarity of a certain method, by which it will be possible for you to get a start to enable you to investigate some of the problems in mathematics by means of mechanics. This procedure is, I am persuaded, no less useful even for the proof of the theorems themselves; for certain things first became clear to me by a mechanical method, although they had to be demonstrated by geometry afterwards because their investigation by the said method did not furnish an actual demonstration. But it is of course easier, when we have previously acquired, by the method, some knowledge of the questions, to supply the proof than it is to find it without any previous knowledge. This is a reason why, in the case of the theorems the proof of which Eudoxus was the first to discover, namely that the cone is a third part of the cylinder, and the pyramid of the prism, having the same base and equal height, we should give no small share of the credit to Democritus who was the first to make the assertion* with regard to the said figure, though he did not prove it." - https://vixra.org/pdf/1111.0111v1.pdf (at page 25)

1 Derivation of Casimir Force

1.1 Physical Mechanism

The Casimir force occurs when two uncharged metal plates in a vacuum are close together. Virtual photons from the quantum vacuum have fewer wavelengths that fit between the plates than outside, reducing the "push" inside and pulling the plates together.

1.2 Step-by-Step Derivation

1. **Vacuum Energy Without Plates**: In free space, photon modes are continuous:

$$E_{\text{out}} = 2 \cdot \frac{1}{2}\hbar \int \frac{d^3k}{(2\pi)^3} c\sqrt{k_x^2 + k_y^2 + k_z^2} = \frac{\hbar c A L}{2\pi^2} \int_0^\infty k^3 dk$$

This diverges, representing the unbounded vacuum energy.

2. **Energy Between Plates**: For plates at $z = 0$ and $z = d$, $k_z = \frac{n\pi}{d}$, $n = 1, 2, \ldots$:

$$E_{\text{in}} = \hbar c A \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} \int_{-\infty}^{\infty} \frac{dk_y}{2\pi} \sqrt{k_x^2 + k_y^2 + \left(\frac{n\pi}{d}\right)^2}$$

In polar coordinates ($k_\perp = \sqrt{k_x^2 + k_y^2}$):

$$E_{\text{in}} = \hbar c A \sum_{n=1}^{\infty} \int_{0}^{\infty} \frac{k_\perp dk_\perp}{2\pi} \sqrt{k_\perp^2 + \left(\frac{n\pi}{d}\right)^2}$$

This sum diverges too.

3. **Energy Difference and Regularization**: Both E_{in} and E_{out} are infinite. We regularize the difference:

Euler-Maclaurin Method: Use:

$$\sum_{n=1}^{\infty} f(n) = \int_0^{\infty} f(x)dx + \frac{f(0) + f(\infty)}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \left[f^{(2k-1)}(\infty) - f^{(2k-1)}(0)\right]$$

where $f(n) = \int_0^{\infty} \frac{k_\perp dk_\perp}{2\pi} \sqrt{k_\perp^2 + \left(\frac{n\pi}{d}\right)^2}$. Substitute $u = k_\perp d/\pi$:

$$f(n) = \frac{\pi}{2d^2} \int_0^{\infty} u\sqrt{u^2 + n^2} du$$

The integral and sum diverge, but corrections (e.g., $B_2 = 1/6$, $B_4 = -1/30$) yield:

$$E = -\frac{\pi^2 \hbar c A}{720 d^3}$$

Zeta Function Method: Define:

$$E(s) = \hbar c A \sum_{n=1}^{\infty} \int_{0}^{\infty} \frac{k_\perp dk_\perp}{2\pi} \sqrt{k_\perp^2 + \left(\frac{n\pi}{d}\right)^2} e^{-s\sqrt{k_\perp^2 + (n\pi/d)^2}}$$

Analytic continuation via $\zeta(-2) = -1/12$ gives the same:

$$E = -\frac{\pi^2 \hbar c A}{720 d^3}$$

4. **Force**:

$$F = -\frac{dE}{dd} = -\frac{d}{dd}\left(-\frac{\pi^2 \hbar c A}{720 d^3}\right) = -\frac{\pi^2 \hbar c A}{240 d^4}$$

Per unit area:

$$F/A = -\frac{\pi^2 \hbar c}{240 d^4}$$

5. **Matrix Representation**: Photon mode density:

$$\begin{pmatrix} \rho_{\text{out}} & 0 \\ 0 & \rho_{\text{in}} \end{pmatrix}$$

where $\rho_{\text{in}} < \rho_{\text{out}}$.